On a Microscopic Representation of Space-Time III

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Using the Dirac (Clifford) algebra γ^μ as initial stage of our discussion, we summarize and extend previous work with respect to the isomorphic 15dimensional Lie algebra su∗(4) as complex embedding of sl(2,$\mathbb{H}$), the relation to the compact group SU(4) as well as associated subgroups and group chains. The main subject, however, is to relate these technical procedures to the geometrical (and physical) background which we see in projective and especially in line geometry of $\mathbb{R}^3$. This line geometrical description, however, leads to applications and identifications of line complexes and the discussion of technicalities versus identifications of classical line geometrical concepts, Dirac's 'square root of p^2', the discussion of dynamics and the association of physical concepts like electromagnetism and relativity. We outline a generalizable framework and concept, and we close with a short summary and outlook.

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I. INTRODUCTION

A. Context so far

In the first two parts ([6], [7]) of this series of papers we've presented a mostly group-based approach to the Dirac algebra where we've started from nothing but very basic assumptions of spin and isospin symmetries in order to describe hadronic observables in the low-energy regime of the particle spectrum. The straightforward part of our approach resulted in a compact SU(4) (A_3) group[1] covering independent SU(2)×SU(2) spin×isospin or isospin×spin transformations, dependent on the respective operator representation (hereafter for short 'rep') identifications.

As the main step, based on several observations, we've introduced *only one physical assumption*: We want to understand this compact SU(4) symmetry, although mathematically represented as an *exact* symmetry, physically as a 'nonrelativistic' (or 'low energy') *approximative* limit of an appropriate relativistic description, so we use compact SU(4) as a (physical) approximation or 'effective' description only in order to use its well-established rep theory of compact Lie groups. With respect to the spectrum, we have to group 'particles' and 'resonances', so consequently we break the (noncompact and compact) A_3 symmetry further by spontaneous symmetry breaking with the Wigner-Weyl realized compact (maximal) subgroup USp(4) and other mechanisms later on. So in [7] we have continued this discussion (see also [6] and references) by presenting some more aspects with emphasis on spontaneously (and later explicitly) broken symmetries and some evidence to relate usual/standard quantum field theory to a background in projective and especially line geometry. Please note once more, that this discussion is *not* restricted to the old (and sometimes simple) spin/isospin hadron interpretation of the reps (see e.g. [1]) but holds for *all* theoretical descriptions based on the Dirac (Clifford) algebra due to its isomorphism[2] with SU∗(4).

In this context, we've begun branching into a parallel thread (see [8] and [9]) which led deeper into projective geometry and transfer principles, and as such to various equivalent representations of geometries (see e.g. [2]). In terms of (Lie) group theory we are thus dealing with the real groups SO(n,m) with $n + m = 6$ and – by complexifying some of the coordinates in use – with various (complex or quaternionic) covering groups and their subgroups. As such we find on various levels correspondences between group transformations and reps on one side as well as geometries and objects on the other side.

What we want to present at this stage of work are some more remarks on physical aspects of a quaternionic projective theory (QPT, see [6], [7] and references therein) and we try to relate them to geometrical concepts. Although at a first glance this seems like rewriting some 'well-known' representations only, in the long term we benefit from a well-defined and unique description in

*Electronic address: dahm@bf-is.de; We thank J. G. Vargas for several interesting discussions during the ICCA-10 conference and especially for pointing out various aspects of Kähler's work as well as Kähler's achievements in differential calculus.

[1] In this energy regime, counting of (grouped) resonances works well with respect to dimensions of SU(4) group reps (see references in [6] and [7]), especially in the PhD thesis Dahm.

[2] We have addressed the problem already that there are various compact low-dimensional symmetry groups which occur automatically in this context. So there is a priori *no need* to introduce manually (and additionally) further degrees of freedom based on such compact groups by hand like in gauge or Yang-Mills approaches. It is more important to gain and to exercise control over the respective (physical) field interpretations [8] introduced already into the Dirac algebra.

terms of line (and Complex[3]) coordinates and their more general justification right from projective geometry by using lines as basic space elements of $\mathbb{R}^3$ or P^3. Last not least lines in the context of tangential and especially tetrahedral Complexe automatically (and naturally) introduce harmonic ratios[4] of points and lines (and as such very naturally metric properties from the viewpoint of Caley-Klein metrics), not to mention polar and conjugation relations and a discussion of second order/class properties. Although here we do not have room to go too much into details of our ongoing work, we want to mention at least some physical relations and contexts of what is our work in progress with respect to electrodynamics and relativity.

As such, in the subsections of this first section we summarize briefly some basic concepts and notations which we need for this presentation. Afterwards, in section II we switch to physical aspects and identifications while in the last section we mention further aspects of the general framework, and we close with a brief summary and outlook of ongoing work.

B. Summary Plücker and Line Coordinates

In order to discuss physics, it is helpful to remember some basic notations[5]. Using four real homogeneous *point* coordinates x_α, $0 \leq \alpha \leq 3$, to denote points in projective 3-space P^3 and by choosing two points x and y incident with the line, we can define the six independent (homogeneous) Plücker coordinates[6] $\mathcal{X}_{\alpha\beta}$ of the line by

$$\mathcal{X}_{\alpha\beta} := x_\alpha y_\beta - x_\beta y_\alpha \quad \text{or} \quad \mathcal{X}_{\alpha\beta} := \begin{vmatrix} x_\alpha & y_\alpha \\ x_\beta & y_\beta \end{vmatrix} \tag{1}$$

where $0 \leq \alpha, \beta \leq 3$. The coordinates are antisymmetric, i.e. $\mathcal{X}_{\beta\alpha} = -\mathcal{X}_{\alpha\beta}$, invariant under common (additive) displacement of both points and they fulfil the 'Plücker condition' $\mathcal{X}_{01}\mathcal{X}_{23}+\mathcal{X}_{02}\mathcal{X}_{31}+\mathcal{X}_{03}\mathcal{X}_{12} = 0$. Moreover, they transform linearly and homogeneously with respect to projective (space) transformations $a_{\alpha\beta}$, i.e. $\mathcal{X}'_{\alpha\beta} = \sum a_{\alpha\mu}a_{\beta\nu}\mathcal{X}_{\mu\nu}$, so that for line coordinates we may use a '6-dim' 'linear' rep p_α, $0 \leq \alpha \leq 6$, with special constraints as well. The second definition in eq. (1) and in general the determinant (re-)formulation on the right – at that time being more of a fashion – is easier to relate to symplectic transformations. By transfer principles, the line and Complex geometry of $\mathbb{R}^3$ can be mapped onto points in $\mathbb{R}^5$ and we can perform analogous (and sometimes easier) point considerations in $\mathbb{R}^5$ where the Plücker-Klein quadric M_4^2 plays an important rôle (for more details see [9]) in that lines in $\mathbb{R}^3$ are points of $\mathbb{R}^5$ located on the Plücker-Klein quadric M_4^2. Investigating images of objects and transformations of $\mathbb{R}^3$ also in $\mathbb{R}^5$, special interest can be given to automorphisms of M_4^2 (see also [9] and upcoming work). The transition to other geometries (like Laguerre, Möbius, spheres, etc.) and related geometrical objects may be performed as well [2]. Another closely related and deeply entangled aspect of line coordinates is Plücker's notion of a Complex (and Möbius' null systems in relation to planar lines of a linear Complex) as well as the related general geometry of Complexe, congruences and Dynamen.

In order to relate to (standard) differential geometry, it is easier to start right from Plücker's (Euclidean) coordinate rep ([23], p. 26, Nr. 26, eq. (1))

$$(x-x'), (y-y'), (z-z'), (yz'-zy'), (zx'-xz'), (xy'-yx') \tag{2}$$

of line (ray) coordinates[7]. If we now (in the sense of continuity and analyticity, or even associating a 'transformation' to 'connect' the two points x and x' involved by using a line segment) require $x'_i = x_i + \mathrm{d}x_i$, i.e. $\mathrm{d}x_i = x'_i - x_i$, antisymmetry of the line coordinates (or the equivalent description by a determinant) provides expressions in terms of coordinates and differential forms which directly lead to line elements $\overline{x'x}$, Pfaffian equations and the *calculus*[8] of differential forms. If in addition we introduce polar relations for the line segment by shifting the point x to be incident with a (second order) surface (i.e. we replace $\mathrm{d}x_i$ by brute force with the tangential 'operators' ∂_i at the original – and then only remaining – reference point x!), we obtain (partial) differential representations of (compact) Lie generators (see e.g. [12] or [13] for their differential rep) according to $x_i\partial_j - x_j\partial_i$. Up to coordinate complexifications (which we'll discuss later briefly), the important fact, however, is the underlying geometry being nothing but line geometry which we can use to describe global/finite geometry, not only infinitesimal issues. Nevertheless, we maintain and we can perform full control over the two points x'_i and x_i from above *individually*, especially also with respect to more advanced projective concepts like polarity,

[3] As before, we have used Plücker's old German notation 'Complex' [23] with capital 'C' to denote line complexes, and as such we have also used the old German plural form 'Complexe'. So mix-ups with complex numbers are (hopefully) avoided, moreover it would be nice to honour this great scientist (although late) by using and establishing at least this small part of his notation.

[4] German: Doppelverhältnisse

[5] A longer derivation of various line coordinates right from the underlying coordinate projections, i.e. starting in terms of inhomogeneous coordinates, can be found in [23]. Take care, however, of the orientation of the underlying coordinate system.

[6] To denote line coordinates we use Study's notation with capital fracture letters.

[7] Plücker usually used (x, y, z) to denote the *coordinates* of one single point p and attached sub- and superscripts to distinguish the points instead of using subscripts/indices attached to points x and y like x_i, y_i, etc. in order to distinguish and enumerate the respective point coordinates.

[8] Note the important fact that we need a calculus to reflect the antisymmetry of the two points $x'_i$and x_i involved, and that in the context of $\mathrm{d}x_i$ we are talking of a calculus only!

conjugation, etc.

For us, it is noteworthy that the coordinate differences $\mathrm{d}x_i$ (see also [9]) on the one hand show well-defined (line) transformation behaviour and a well-established geometrical interpretation as projections, on the other hand typical 'coordinate' transformations $\delta x_i = x'_i - x_i = \mathrm{d}x_i$ can be mapped to known Lie algebraic transformation concepts like $\delta_Y \sim [Y, \cdot]$ or to transformations of differential forms using a 'transferred', adopted and adapted geometrical calculus applied to those differentials. As such, also advanced algebraical and analytical concepts of such calculuses can be re-transferred back to (projective) geometry[9] and especially to line transformations and line geometry. So we think line (and Complex) geometry is much better suited to describe physics in terms of global/finite mathematical concepts than the various concepts 'derived' from infinitesimal/differential geometry only.

C. 'The Metric'

We have mentioned already [9] the mixture in notion nowadays when working with vectors as well as the sometimes misleading (and most often 'vector-derived') notion and use of a metric. In most cases a 'vector', although formally a coordinate *difference* (or 'the upper half' of the line rep in (2)), is used by setting one of the two points to coincide with the origin 0 of a 'coordinate system'. This shrinks the coordinate difference to single point coordinates only which afterwards often spoils the concept. The notion 'metric' – whether in the usual Euclidean sense or in the framework of (semi-)Riemannian spaces – usually describes a symmetric (and most often diagonal) structure which is used to 'contract' two objects (usually vectorial or tensorial reps) which themselves transform linearly. In most cases this notation is nowadays used in conjunction with linear reps (on spaces/modules), and it is a fashion to discuss low-dimensional rep dimensions in the beginning and generalize soon to arbitrary (and sometimes infinite) rep dimensions. Typical examples are space-time using x_a, $a \geq 4$, and the related dynamics in various formulations, usually based on related momenta p_a, $a \geq 4$, when applying Hamiltonian dynamics or 'quantum' approaches, and even 'time'-associated Lagrangian concepts and (partial) differential equations. It is often overseen when starting from coordinates only and counting the coordinates naively, that already switching the coordinate *interpretation* changes the 'dimension' of such objects or of the underlying rep space. Simple examples are e.g. given by the 5-dim coset space p (or $\exp p$) when switching from 'space-time' (point) interpretation as usual in nonlinear sigma models (or SSB models) to (infinitesimal) line elements (Lie), lines, Complexe or even more sophisticated geometrical models[10]. It is often also overlooked or ignored that conic sections and cubic or higher order curves are projectively generated or have intrinsic relations to other objects in projective geometry, e.g. planar curves to linear Complexe and Complex lines in the plane [23].

This eclipses the fact that in order to perform physics we identify observable objects with special mathematical reps, and we map their (transformation) behaviour to reps having finite dimension only[11]. The same holds for well-established projective concepts like polarity, conjugation and duality whose interpretations when associated with physical objects are often messed up by a generalization to arbitrary dimensions although one is – at least sometimes – still able to define a similar formal calculus and perform some calculations 'for arbitrary dimension n'. It should be mentioned here that it is often the background of projective (or incidence) geometry within its application and usefulness for logic (see e.g. [11] with respect to 'abstract geometry', there especially appendix II) which still provides the helpful axiomatic background.

Here, we restrict our discussion to $\mathbb{R}^3$ where we have 3- or 4-dimensional (linear) point reps, dependent on whether we use inhomogeneous or homogeneous/projective coordinates. With this coordinate interpretation already the line reps may have 'dimension' 4, 5 or 6 [23], and we know from the very beginning of projective geometry, from duality, from the projective construction of objects (e.g. conic sections) or more general from synthetic geometry that we can switch from using orders to using classes and thus interrelating dimensions. So using P^3 as well as simple and well-known geometric objects, we are far from using only 3- or 4-dim reps to describe space-time objects and behaviour, and we find much more symmetry structure than simple transformation groups like SO(3) or the Poincaré group only [9].

On this footing, interpreting the momentum $\vec{p}$ as (polar part of a) line rep and $\vec{x}^2 = r^2 = (ct)^2$ as a sphere with (infinite) radius per given common time 't' for all (projective!) space-dimensions x_i and x_0, it is natural to (re-)introduce line coordinates as a unifying description which automatically comprises 'non-local' effects. The only price we have apparently to pay is a loss of the direct (physical) coordinate interpretation of x_α as well as a loss of naive 1-dim parameter differentiation within the general geometric approach to describe point trajectories and/or orbits. This 1-dim and mostly differential geometric aspect, however, can be recovered by transitioning from (general) lines to line elements while restricting the

[9] We thank J. G. Vargas for pointing us to Kähler's work (see e.g. [15]) which we find really interesting to study in more detail also in the context of line geometry.

[10] See e.g. [11], appendix II on 'abstract geometry' related to dim 5!

[11] i.e. the reps depend on a finite number of parameters only!

geometry and coordinatizing the respective geometrical setup by appropriately chosen (inhomogeneous) coordinates. Moreover, this is just what Lie did when establishing 'Lie algebras' and the differential rep of generators[12]. So we feel free to work with line geometry (or in some places even with the fully-fledged framework of projective geometry) and we want to see how we can describe physics.

In this context, it is natural to understand the 'norms' $\vec{p}^2$ and $p^2 = p_\mu p^\mu$ in terms of a *square* of (a part of) a line rep and as such – remembering self-duality of lines in $\mathbb{R}^3$ – when linearizing such squares we have to end up with a (linear) 5- or 6-dim line rep instead of a 3-dim 'vector' only[13]. So in the standard treatment using the (3-dim) 'vector' approach only, parts of the momentum rep (and as such moments and parts of the energy) are often missing and are not considered in calculations. The same holds for bilinear representations of a 'metric' in order to linearize quadratic objects like in Clifford algebras, on semi-Riemannian or even 'two-point' homogeneous spaces which introduce 'the metric' only with respect to the point part or the point difference of the points of the respective manifold(s), i.e. 'the upper half of the line rep' given in eq. (2). A naive generalization in terms of arbitrary (point) dimensions spoils the line as well as the physical background, i.e. although formally we can rewrite (in Euclidean interpretation or using the four-vector calculus of special relativity) $p^2 = m^2$ in terms of a linear rep p and a symmetric formalism $\{\gamma_i, \gamma_j\} = \delta_{ij}$ or $\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu}$ (see e.g. [1] or [20], ch. 1-3), the simple (formal) abstraction of a metric is algebraically nice to handle but too simple in order to highlight the complete geometrical (polar) background of such an 'anticommutator'. Of course one finds an appropriate algebraic and analytic calculus and a generalization to arbitrary n with lot of nice group theory attached but – as history proves – the fact that 6-dim line reps can be composed of two 3-dim 'vectors' ('polar' and 'axial') and as such exhibit naturally a SO(3)×SO(3) transformation structure seems forgotten nowadays. Moreover, 'the lower half of the line rep' (2) being apparently of 2^{nd} order in the point reps has a *physical* identification by itself in terms of moments!

Even worse, allowing for individual coordinate complexifications (as long as we preserve the real 'norm' constraint $v_i^2 = \text{const}$) for both of the two 3-dim 'vectors' v_i of the line rep (2), we can as well discuss SU(2)×SU(2) or twofold quaternionic transformations U(1,$\mathbb{H}$)×U(1,$\mathbb{H}$) acting on these constituents but now we know the reason for the different polar and axial behaviour of the constituents being an artefact of the Euclidean point rep used in (2) to represent the line by two Euclidean 3-dim 'vectors' v_i. So the discussions of chiral symmetry and chirality fade out in the light of this background as being governed and superseded by lines and screws when complexified appropriately, i.e. being transferred to elliptic geometry. This outlines our intention and motivation to revive line and projective geometry instead of following the usual 'linearization' of $p_\mu p^\mu = m^2$ by[14] $p_\mu \gamma^\mu$ discussing 'quantum' 'anything' and attaching algebra and analysis naively in form of one or the other calculus attached to points or naive (point) manifold concepts. For us – argueing in Plücker's sense[15] – the difference is the necessary switch towards using lines instead of only points (even if accompanied by planes) as the underlying base elements of space where people perform all kinds of analysis in 'space-time', even in terms of very sophisticated concepts of differential geometry (see e.g. [21]) which – in our opinion – hide more physics behind formal mathematics than they are able to show or describe.

[12] We think that this achievement is tightly related to having got knowledge on Plücker's work and establishing intensive contact to Klein after having met Klein for the first time in October 1869 during Klein's 'Berlin time' from August 1869 to March 1870. However, we want to leave the (more) complete and final discussion and judgement to science historians.

[13] Thanks to his talk and private communication with O. Conradt during the conference, we heard that Dirac knew much about projective geometry, and that it was Dirac who searched for (algebraic) reps of his results from within projective geometry. However, we do not have access to those references yet.

[14] We just want to remember the fact that this equation is independend from the mass as m drops out. Indeed, we see this as an equation for 4-velocities u_μ describing the velocity constraint $u_\mu u^\mu = 1$.

[15] It is to be pitied that the enormous achievements of this great scientist are not only not honoured but even almost forgotten. To top this deficit, even his own university was able to publish only a short note [3] in order to remember his 140th anniversary of death in 2008. But even there, people put more focus on his CV and his 'strong' and 'own' personality than on his enormous achievements in mathematics and physics (see e.g. [4]). Indeed, a lot of Plücker's results were absorbed later in Lie's, Klein's, Clifford's and Ball's work mentioning Plücker only in general or even without citing or mentioning Plücker at all. This might be attributed to the fact that Plücker inbetween worked for decades in physics (and especially optics) only before returning to mathematics while teaching and advising Klein in physics and mathematics. It was Klein in conjunction with Clebsch to summarize at least some of Plücker's late and more systematic results on line geometry [23], based on existing manuscripts and on the outline originating from Plücker while Plücker himself only had time to publish two late presentations in 1865/66 on generalizations of lines to 'Complexe', 'Dynamen' and their tremendous use for physics before his death in 1868. For example, the treatment of oval surfaces in relation to generating line sets can be found in [19] (see e.g. ch. II, §§4–6). Some very powerful consequences with respect to dynamics, differential geometry and cones have been given and appreciated by Clebsch in [5]...

D. Summary 'Spheres' and Complexe

The most important aspect in our current context[16] is the transition from typical 'light cone' reps $\vec{x}^2 - x_0^2 = \vec{x}^2 - (ct)^2 = 0$ to lines and the transformation of this constraint. Note already here that this framework can be applied also to point reps not on the light cone (or in 'momentum space' for 'massive particles' 'on the mass shell') by generalizing lines to 'Complexe', 'Gewinde' and null systems, 'Dynamen', 'Somen' or screws (see e.g. [26] and references therein). Whereas most usual treatments assume 'affine' point coordinates x_α in Minkowski's four-vector notation, we have already pointed out (see [6] and [8]) that for same/equal 'time' t in all four coordinates x_α, the coordinate *value* ct related to the coordinate x_0 has to be treated as infinity (∞) which can be done in (four) homogeneous/projective coordinates and the framework of projective geometry only[17], *not* in affine or Euclidean coordinates (see e.g. [19]). The appropriate rep of space (point) coordinates, $x_i = v_i t$, to achieve an equally parametrized footing thus automatically introduces parameters $\beta_i = v_i/c$ by using a (projective) Cayley-Klein metric when switching to inhomogeneous/affine (point) coordinates[18]. The parameters β_i which appear in physical transformations thus turn out as a reminiscence of line geometry while using inhomogeneous coordinates $x'_i \sim x_i/x_0$. Although being – in conjunction with points as basic space elements – THE backdoor of Newtonian ideas and concepts within four-vector calculus, a parameter 'time' allows people to express dynamics by performing differentiation while sticking to the point picture and its related dynamical concepts whereas part of the discussion can be mapped to velocities and their relations as is typically done in special relativity. But special relativity (see section II B) can also be interpreted in terms of line and Complex geometry easily, and we use the individual/local times 't' and 't'' of two coordinate systems only to select the respective subsets of lines out of all lines comprized within the geometrical setup. So the task of (local) coordinates in a sense is to relate and group certain lines in a large overall 'line set' of a common geometrical setup. In other words, we can use 'times' to group and sort lines or aggregations of lines (and related objects like points, sections or higher order/class curves) within the dynamical behaviour of the setup. Especially 'features' like the invariance of normal planes (i.e. $x = x'$, $y = y'$ while translating along the z-axis) known from Lorentz transformations thus have straightforward geometrical background from Complex geometry and null systems.

Using a parameter $\epsilon^2 = \pm 1, 0$ to describe the respective non-Euclidean and Euclidean geometries, the transition of 'light cone' reps $\epsilon^2 \vec{x}^2 + x_0^2 = 0$ in terms of (4-dim) point coordinates x (or $\epsilon^2 u_0^2 + \vec{u}^2 = 0$ in terms of (4-dim) plane coordinates u) into a line rep in terms of six related homogeneous line coordinates $\mathcal{X}$ is known to be performed by

$$\mathcal{X}_{01}^2 + \mathcal{X}_{02}^2 + \mathcal{X}_{03}^2 + \epsilon^2 \left(\mathcal{X}_{12}^2 + \mathcal{X}_{23}^2 + \mathcal{X}_{31}^2\right) = 0 \qquad (3)$$

or in the more symmetric form

$$\frac{1}{\epsilon}\left(\mathcal{X}_{01}^2 + \mathcal{X}_{02}^2 + \mathcal{X}_{03}^2\right) + \epsilon \left(\mathcal{X}_{12}^2 + \mathcal{X}_{23}^2 + \mathcal{X}_{31}^2\right) = 0$$

which simplifies Euclidean geometrical reps and discussions. Whereas the general theory necessary for physics mounds at least into the framework of quadratic line Complexe[19], here we want to mention only the fact that the lines of a Complex of nth degree if they are incident with one point (resp. they meet in one point) of $\mathbb{R}^3$ constitute a conic surface of nth order ([23], §2, p.18), and the lines envelop a planar curve of nth class. So quadratic Complexe constitute a cone of second order in space meeting in one (or each) point as required by [10] which we have associated with 'the photon' (see also [7]), and we can study associated planar conic sections of second class which we can relate to (quadratic) invariants.

The limit towards Euclidean geometry has to be performed carefully. However, in this limit we find from above the constraint $\mathcal{X}_{01}^2 + \mathcal{X}_{02}^2 + \mathcal{X}_{03}^2 = 0$ involving the x_0 coordinate(s) of the point rep(s). Besides switching between Plücker and Klein coordinates, we can complexify further (individual) coordinates so that in general we have to discuss related transformation groups SO(n,m), $0 \le n, m \le 6$ with $n + m = 6$ or the related complex transformation groups SU(n,m), $0 \le n, m \le 4$ with $n + m = 4$, or even quaternionic transformations like Sl(2,$\mathbb{H}$) (or SU$*$(4), respectively). Dependent on the inertial index[20] (or signature) of the quadratic form (3),

[16] Please note, that the expression $p := \vec{x}^2 - r^2$ is known as 'potency' (German: 'Potenz') of spheres and that we may branch here to sphere Complexe and their geometry [25] as well. That's, however, beyond the current scope of presentation here (see e.g. [9] with respect to transfer principles) although there are 'tons of' very interesting applications of this representation scheme in physics. What we also don't want to discuss here in more detail is the interpretation of special relativity in terms of such sphere 'invariances' in different coordinate systems and with the additional constraint $x' = x$ and $y' = y$ or $\mathrm{d}x\mathrm{d}y = \mathrm{d}x'\mathrm{d}y'$ in the normal plane. Therefore we need much deeper background with respect to sphere Complexe and Complex geometry.

[17] Please note, that this has to be discussed very carefully in terms of coordinate values and (binary) parameters, and care has to be taken in identifying homogeneous and inhomogeneous coordinates and their respective coordinate values/projection parameters. Moreover, an exact description related to Euclidean geometry has to be performed using normal instead of Euclidean coordinates which square to 'infinite radius' as they approach infinite values, i.e. for absolute elements/points.

[18] Please note the intrinsic assumption of Klein's Erlanger program by assuming t to parametrize the transformation(s) thus connecting the transformation parameters to velocities of an Euclidean, affine scenario!

[19] Quadratische Complexe

[20] German: Trägheitsindex

we can of course define linear reps and a 'metric' for a 'norm' being invariant under the respective SO(n,m) symmetry group, $n + m = 6$; SO(3,3) and SO(6) for Plücker and Klein coordinates are well-known, subsets und 'sub'-symmetries are discussed in [2]. The general form $a_{\alpha\beta}\mathcal{X}_{\alpha\beta} = 0$, $a_{\alpha\beta} \in \mathbb{R}$, defines a (linear) Complex $\mathcal{A} \sim a_{\alpha\beta}$ or a 6-dim real 'vector' respectively, and we can distinguish singular and regular Complexe and apply the framework of Complex geometry and symplectic symmetries.

Last not least, in this context we want to mention one more aspect of our ongoing work in that Plücker has associated Complexe (resp. lines and axes) and especially congruences of two or more Complexe to ellipsoids (see [23], 'Erste Abtheilung', §3, p. 99ff, ibid. §3, eqns. (46)ff or [23], 'Zweite Abtheilung', preface and main text) or various more general types of surfaces. There is indeed much older work [22] where Plücker defined such specialized ellipsoids in the context of Fresnel's wave theory, confocal surfaces and 'potential theory'. For us, this provides some geometrical background of the nowadays usual and common mystification of the 'wave-particle dualism'. Plücker (and other people at that time) knew well that while working with Complexe and (some of) their congruences, one finds line reps with naturally associated ellipsoids [22], and (strictly) spherical problems like Laplace or Schrödinger equations are special cases only. The separation denoted nowadays as a 'dualism' is caused by describing 'point' particles by only half (i.e. the polar part) of the originally necessary line rep (2) while sometimes playing strange games with Euclidean/affine dynamics. So instead of mystifying the relation and interconnection of the two descriptions, one should think in terms of lines and transfer principles to resolve such 'questions'.

Due to a line being a priori free in $\mathbb{R}^3$ (or P^3) to connect a point with an observer (i.e. always by its very and axiomatic definition to connect at least two points), we can a priori handle (space-related) 'extension', different coordinate choices by investigating and/or transforming the fundamental tetrahedra and 'non-localities' especially of 'the photon'[21]. Tangential spaces are special cases of polar setups in conjunction with conics or surfaces which themselves can be treated in general and complete by projective construction mechanisms and the discussion of 'class' instead of 'order'. We can use the important apparatus of tangential and tetrahedral Complexe (see e.g. [27], [24]) and moreover we have a 'natural' definition of conjugation right from geometry. Last not least, invariance of a line under transformations automatically provides (affine) translation invariance when expressed in point coordinates so with respect to the Poincaré group and contractions we definitely have a well-defined geometrical framework which can be treated by lines or 'Gewinde' and geometrical limites thereof [28], [26].

As an example, after having accepted line coordinates and line reps, one can easily apply incidence relations of lines in (6-dim) line coordinates[22] p_α, $1 \leq \alpha \leq 6$, and work e.g. with Klein coordinates[23] in order to relate equations like $\sum p_\nu p_{\nu+3} = 0$ or $\sum x_i^2 = 0$ to the framework of ruled surfaces (see [28], Vol. 2, I §4). This facilitates a direct generalization to Complex geometry.

II. PHYSICAL IDENTIFICATIONS

As this is ongoing work and room is short here, we'll mention briefly some aspects of identifying physics with such geometrical concepts.

A. Electrodynamics

We've argued already (see [7]) within the framework of spontaneous symmetry breaking (SSBs) that we want to use a Goldstone identification of the (massless) photon in SU$*$(4)/USp(4) in order to relate equivalence classes of velocities and the 'masslessness' of photons in common QFT frameworks. The physical equivalence is the connection of velocity changes (in the coset) with photon emission ('Bremsstrahlung'), and as a consequence we relate redefinitions of USp(4) Wigner-Weyl reps and especially the ground state to photon emission resp. (gauged) energy changes. Although this is reasonable from the physical viewpoint in that we can relate (hard) observations to such models, the mathematical and physical formulations using differential geometry at the one or other point look hazy. So people introduce 'velocity' 4-vectors k^μ 'on the light cone' and 'polarizations' ϵ^μ with additional constraints in an 'affine' interpretation which lead to the one or other obscure explanation or philosophy. In this context one can mention conditions like the 'masslessness' (of 'particles') $k^\mu k_\mu = 0$, the distinction of 'on-mass-shell' and 'off-mass-shell' behaviour (or 'virtual particles') in interaction processes and 'gauge conditions' like $k^\mu \epsilon_\mu = 0$ or even $\vec{k} \cdot \vec{\epsilon} = 0$, i.e. 'orthogonality', in conjunction with using normals of normals like with $\vec{k}$, $\vec{E}$ and $\vec{B}$.

For us, the problem to determine a ('vectorial' and 'affine') 'velocity' k^μ as a physically meaningful, linear dynamical object ends in front of the fact that light by definition spreads out 'on the light cone', i.e. by construction on a second order (null) cone with the maxi-

[21] The discussion of relating differential geometry to projective geometry has been a major topic for decades around the turn of the 19th to the 20th century. However, the assumptions, specializations and drawbacks introduced into differential geometry and calculuses seem to be forgotten…

[22] German: Plückersche Zeiger

[23] German: Kleinsche Zeiger

mum (and for 'massive particles' unreachable, i.e. *infinite*) 'velocity' in order to transport information. As such we can honestly derive this spreading from a construct with $\vec{k}$, $\vec{\epsilon}$ and the Poynting vector (i.e. from two 3-dim objects $\vec{E}$ and $\vec{B}$ respectively $\vec{H}$ related to the physical force $\vec{F}$) only while keeping in mind that in order to treat this type of infinity 'on the light cone' we have to use homogeneous coordinates! So we can use Klein's remark (see [16]) that (for homogeneous coordinates!) the Plücker condition $\sum p_\nu p_{\nu+3} = 0$ is sufficient to define a line (rep). The general way to solve this problem is to use line (or Complex) coordinates.

However, for us that's not really sufficient because we are not only working with simple lines or linear Complexe but also with quadratic ones (or at least with quadratic constraints using linear Complexe). Moreover, we know that electromagnetic *forces* related to $\vec{E}$ and $\vec{B}$ are to be described via the Lorentz force, and that in Hamiltonian (and also in Langrangian) formulations of dynamics we can start using $\vec{E}$ and $\vec{B}$ in terms of the antisymmetric field strength $F_{\mu\nu}$ although nowadays people prefer to use the description via the potential(s) A^μ and partial derivatives thereof, mostly as a trade-off to a Lorentz covariant description and differential reps. Whereas the rep of the 'potential' A, as dependent of k and ϵ, can be naively related to a line rep comprizing $\vec{k}$ and $\vec{\epsilon}$, at the same time we have to take care of the two normals $\vec{E}$ and $\vec{B}$ and their dynamics, too.

Now a major point of discussion for us at the moment is a possible identification of the tensor $F_{\mu\nu}$ with a line rep (or a linear Complex). The 'tensor' character of this object (with *two* indices) is caused formally only by Minkowski's four-vector formalism. We can ad hoc associate the space components of F_{ij} (the (Euclidean) vector components of $\vec{B}$ (or $\vec{H}$)) with the axial part of the 6-dim line rep, and we can associate the components F_{0i}, i.e. the components $\vec{E}_i$ (see e.g. [14], ch. 11), with its polar 3-dim part. Then the orthogonality relation $\vec{E}\cdot\vec{B} = 0$ may simply be interpreted as the Plücker constraint to fulfil the line condition although the association of a polar 3-dim vector rep with null-components in the face of eq. (3) and its Euclidean transition seems to be not the best choice of identification. And yes, we have to talk about six *homogeneous* line coordinates which makes it difficult to interpret $\vec{E}$ and $\vec{B}$ directly in terms of physically observable or measurable objects but we have to keep in mind that also the charges (as well as the masses) are only defined *in relation* to another charge (or mass) as is known from Coulomb's (and Newton's) law[24]. The discussion of Lab measurement brings us back to discuss the introduction of (local) time 't' like in $\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p}$ or $\vec{F} = m\vec{a}$ (indirectly).

Whereas we can use products like $F^{\mu\nu}F_{\mu\nu}$ to represent squares[25], our investigations especially in the context of Complexe and Complex congruences have started only. So as ongoing 'program', we have to map physical observations (i.e. objects and their dynamics!) to Complex geometry[26].

With respect to electrodynamics, the introduction of the 'dual' 'tensor' $\mathcal{F}^{\alpha\beta}$ via $\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ enhances the scenario and introduces further aspects into the Complex representation[27]. From the viewpoint of line or Complex geometry, this 'new' object reflects advanced (algebraic) operations of a 6-dim line calculus in that we have to treat line incidences, i.e. 'products' of line reps or parts thereof which resemble inner 'vector' products or 'norms'. So the 'skew tensor' approach corresponds directly to (6-dim) line geometry, and products of (skew-symmetric) 'tensors' are able to represent (6-dim) multiplications in line coordinates, i.e. lines and incidence relations of lines. We'll have to transfer and extend this to products of Complexe and/or an appropriate Complex geometry and its consequences. So at a first glance, line geometry works pretty well for electromagnetism in order to cover the four-vector formalism. What is under construction (or 'open') at the time of writing, is the association between algebraical and physical objects and a deeper understanding of line congruences[28] as well as the physical meaning/identification of $\vec{E}$ and $\vec{B}$ versus $\vec{k}$ and $\vec{\epsilon}$. There are indeed more sophisticated than simple line concepts. If we associate the 3-dim 'field' reps to (linear) Complex parameters $a_{\alpha\beta}$ which (due to Cayley) can be interpreted as line coordinates fulfilling the Plücker condition, too, if $a_{\alpha\beta}p_{\alpha\beta} = 0$ and $p_{\alpha\beta}$ are line coordinates of incident lines[29], for (six) linear Complexe, a constraint formally similar to the Plücker constraint can be formulated as well to construct a quadratic Complex (see [16], Nr. 26).

A further extension of the Complex identification

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[24] This results also from Plücker's identification of forces with respect to line reps, see references in [23]. So in experiments we expect to see charge and/or mass relations like reduced masses or physically observable combinations like e/m only which emphasizes the physical formulation by the Lorentz force when describing dynamics and (Lab) measurable 'accelerations'.

[25] And as such energies! According to our current understanding, that's the reason why the electromagnetic description works well on the classical as well as on the quantum level using Hamiltonian/Lagrangian formulation.

[26] This is in some parts not new but the problem is that science industry today uses (although limited in a lot of aspects) all kinds of 'vectors' or linear reps and not line or even projective geometry, and a lot of old knowledge is simply forgotten in favour of algebraic and analytic technicalities around all kinds of linear vector spaces.

[27] However, we do not want to discuss transitions from ray to axis line coordinates and the related duality considerations of points and planes in $\mathbb{R}^3$ here.

[28] Especially also with respect to the identification of 'ray systems' (German: 'Strahlensysteme erster Ordnung und erster Classe') and their geometry.

[29] German: Treffgeraden

is based on Complex geometry if we go back to the second order surface given in eq. (3) while choosing $\epsilon^2 = -1$ and if we invoke polarity. Then the two Complexe $C_1 = (\mathcal{X}_{01}, \mathcal{X}_{02}, \mathcal{X}_{03}, \mathcal{X}_{23}, \mathcal{X}_{31}, \mathcal{X}_{12})$ and $C_2 = (-\mathcal{X}_{23}, -\mathcal{X}_{31}, -\mathcal{X}_{12}, \mathcal{X}_{01}, \mathcal{X}_{02}, \mathcal{X}_{03})$ are polar with respect to the surface, and we can start applying further reasoning from Complex geometry and compare to physical observations.

B. Special and General Relativity

In order to extend what we have said above to 'relativistic' physics, the simplest approach is to include observers right from beginning into the mathematical description. This, too, is automatically provided using line geometry. If we imagine for a moment the simplest scenario of an observer at rest watching a (non-accelerated) moving point (in some distance), then – as time elapses – we have at a first glance a point (the origin of the observer at rest) and the line of the (moving) point, i.e. a collection of points at different (observer) times, of course, or with different 'coordinates' parametrized by time. This scenario can, of course, be interchanged by choosing an arbitrary point on the line of the moving point to be at rest, or by assuming both points moving (thus introducing a third point at rest not incident with each of the lines). If the two lines of the moving points don't meet we obtain a geometrical (global) setup of two skew lines which we can use to construct a global geometrical picture with times and space coordinates being a subsidiary concept only while mapping the two lines. This illustrates explicitly that if we use the 'physical' information of the relative velocity of the two points to parametrize this scenario, the notion of time – whether from the observer's or the moving point's side – is needed to parametrize the individual coordinate notations and definitions only. But moreover we find a (planar) set of lines connecting the observer's point in space with points on the line representing the trajectory of the linearly moving point. Now, if in addition we allow for the observer to move[30], the trajectory of the observer is a line, too.

If we now play the same game as before by connecting points on the two lines, velocities connect the individual 'line times' to causality in the respective (local) coordinate systems and their related description(s) of physics. On the one hand, as such we can introduce and use individual (point) coordinate systems or apply the typical reasoning of special relativity in terms of point (or four-vector) coordinates. On the other hand, the picture of projective and especially line geometry offers some old and well-established frameworks to describe such a setup much better. We have mentioned already the idea to understand original skew lines from above as axes of (singular) Complexe and the lines intersecting both original lines as a linear congruence. So we can also map the two coordinate identifications to the respective coordinate systems. But besides being still free to use individual coordinate systems related to each line (e.g. in associating the 6 line coordinates to the sides/lines of a (fundamental) tetrahedron individually), we can use in addition the two lines of the trajectories (moving observer and moving point) as opposite sides of a (third) tetrahedron and introduce 'overall' coordinates (with an additional unit point and (if necessary) absolute elements or by associating the framework of tetrahedral Complexe) in order to establish a common description/coordinatization of both systems. So we'll have to work out the algebraic relations of the respective six line coordinates of the two individual line identifications used to describe the two individual coordinate identifications versus using a common (fundamental) tetrahedron related to a parametrization by relative velocity and an abstract overall time which will result in identifying point sets of line incidences and harmonic ratios and relating them while respecting (some or all) properties of projective transformations. This reminds correlating one-particle rep descriptions in quantum field theory (QFT) in order to find common and comparable physical behaviour like in Smilga's nice work (see [8] and Smilga's reference). Moreover, we can 'collect' all the lines connecting the two trajectories (at different (individual) times and as such space points, of course) and describe them via line incidences[31] of both lines, or more general in a first step by singular Complexe[32], ray systems (see footnote before) and by appropriate congruences. This can be done not only in Euclidean geometry but the framework of line geometry (because imbedded in projective geometry) is available also for all types of non-Euclidean geometries which is necessary due to various coordinate complexifications relating the covering groups of SO(m,n), $1 \leq m+n \leq 6$. The physical picture of such a description becomes transparent and clear if in mind we associate a 'light' source to both the moving point and the observer, and if we think in terms of rays being emitted by the point and by the observer, respectively. Nonlinear movements can then be described by e.g. higher order (or higher class) curves and surfaces, and projective geometry provides construction mechanisms, dimension formulas and a lot of further useful tools. Thus, the 'physics' or dynamics is directly related to the geometry of the respective curves and/or trajectories. As mentioned above, the breakdown to (squares of) line elements $\mathrm{d}s^2$ is possible in various ways and respecting/representing various geometries and associated symmetry groups.

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[30] At first, we assume non-accelerated movements and skew/non-incident lines of observer and point.

[31] German: Treffgeraden

[32] German: Treffgeradenkomplexe

III. OUTLOOK

Having in mind how we have associated (physical) light with 'light cones' above, we have additional possibilities on a linear representation level to generalize lines (i.e. singular Complexe) to general linear (and higher degree) Complexe and, moreover, we can investigate their relation to 'massive' reps 'on' and 'off' the mass shell. So what is open today besides a thorough and complete analytical framework is an a priori explanation of the Hamiltonian structure (and as such the energy) of being quadratic in line coordinates[33]. As such, the generalization (and also our ongoing program if we think on how to approach general relativity) is twofold: We can extend the use and application of Complexe and Complex geometry, and we can investigate their various constraints with respective mappings to physics.

Because differential geometry (by using forms linearly) a priori reflects only (polar) parts of line reps and affine behaviour, we are convinced to find additional energy-momentum contributions to $T^{\mu\nu}$ (in four-vector notation) by simply taking elements of line or Complex reps (e.g. moments) into account or even more sophisticated mechanisms of line or Complex geometry and especially higher order Complex geometry. Moreover, we see differential forms, Pfaffian equations (one-forms) and Lie theory as subsidiary concepts of line geometry only in their respective (inhomogeneous) 'time' dependent limits. Using line geometry, the inclusion of observers is a priori guaranteed by the formalism, i.e. we do not have 'observer-free' physics as nowadays usual, and there is a priori no more need to speculate on 'non-localities' because lines *are* non-local and their reps respect this property by 'the lower half' of (2). Last not least, with respect to dimensions of transformation groups and concerning reality conditions so far, we want to mention the possibility that in choosing a 'correct' set of coordinate reps we may associate the 15-dim resp. 16-dim transformation groups with projective transformations mapping (linear) Complexe to Complexe and the 10-dim subgroup mapping null lines[34] to null lines. However, that's an open issue right now and has to be proven formally.

At the time of writing, we see various still 'competing' possibilities[35] to work with 5-dim p (or $\exp p$) and identify the space physically (better: dynamically!), and we feel the need also to discuss the double 15-dim (automorphic) collineations of M_4^2 with respect to physics and real/complex descriptions much deeper in those contexts. This is especially interesting when starting from Hamiltonian formalism while using/identifying line coordinates and assuming the quadratic structure as originating from M_4^2 as the fundamental form (see [17]). We have given above already one application of polarity, however, we have to investigate the physical consequences in much more detail. We have mentioned as well tangential complexes and congruences which we have to arrange versus the current concept of affine connections (see e.g. [18]).

Another open problem is the identification of Complex-related numbers and/or constants versus reps and rep dimensions in the Lie group/algebra related approach, i.e. there are 15 constants mapping the Plücker constraint linearly onto itself, there exist polar systems depending on 20 constants (and series thereof as well) which map (arbitrary) lines to linear Complexe and vice versa, etc. For the low-energy regime of the spectrum, we have associated already an SU(4) interpretation in terms of observable SU(2) spin and isospin degrees of freedom as an approximate compact description, however, being the most compact real form we have to map this to line geometry and dynamics using transfer principles and appropriate coordinate complexifications.

We are convinced [8] that various low-dimensional Lie groups and algebras, especially su(2)⊕u(1), occuring in various applications of QFT are artefacts of certain aspects of line (and projective) geometry of $\mathbb{R}^3$ which emerge by taking and generalizing certain analytic and algebraic aspects of line (and Complex) reps only in terms of individual 'calculuses' and 'rules'. As such we see Plücker's M_4^2 and the twofold 15-dim automorphisms in a central rôle, governed however by the rules of projective geometry. In this context, there are lots of further deep geometrical connections to other topics like Kummer's surface, Darboux's 5-dim reps of confocal cyclids[36], Pasch's sphere Complexe and their geometry or to rep dimensions occuring both in line/Complex geometry and and physical/QFT rep identifications which we have to work on.

[33] Right now we can conjecture only that this quadratic form is related to the fact that the line representation of $\mathbb{R}^3$ is four-dimensional and we need a (quadratic) constraint to eliminate one degree of freedom/dimension. There are further (quadratic) explanation possibilities originating from tangential and tetrahedral Complexe or from the M_4^2 above. The possibility that a second order description of energy itself is an approximation only and that we have to treat this question based on general homogeneous functions is beyond scope at this time of writing.

[34] German: Nullgerade

[35] Here, we want to pinpoint once more [11], appendix II, this time in the context of mapping his two abstract spaces S and S' by collinear mappings.

[36] German: Konfokale Zykliden

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